# Widths of weighted Sobolev classes with weights that are functions of distance to some $h$-set: some limiting cases

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## 1 Introduction

Let $X$, $Y$ be sets, and let $f_1,\, f_2 : \; X\times Y \to \mathbb{R}_+$. We write $f_1(x,\, y) \underset{y}{\lesssim} f_2(x,\, y)$ (or $f_2(x,\, y) \underset{y}{\gtrsim} f_1(x,\, y)$) if for any $y\in Y$ there exists $c(y)>0$ such that $f_1(x,\, y) \leqslant c(y)f_2(x,\, y)$ for any $x\in X$; $f_1(x,\, y) \underset{y}{\asymp} f_2(x,\, y)$ if $f_1(x,\, y) \underset{y}{\lesssim} f_2(x,\, y)$ and $f_2(x,\, y) \underset{y}{\lesssim} f_1(x,\, y)$.

Let $\Omega\subset \mathbb{R}^d$ be a bounded domain (i.e., a bounded open connected set), and let $g,\, v:\Omega\rightarrow (0,\, \infty)$ be measurable functions. For each measurable vector-valued function $\psi :\; \Omega\rightarrow \mathbb{R}^l$, $\psi=(\psi_k)_{1\leqslant k\leqslant l}$, and for any $p\in [1,\, \infty]$ we set

$$\|\psi\|_{L_p(\Omega)}=\Big\|\max_{1\leqslant k\leqslant l}|\psi_k|\Big\|_p=\left(\int\limits_\Omega \max_{1\leqslant k\leqslant l}|\psi_k(x)|^p\,dx\right)^{1/p}.$$

Let $\overline{\beta}=(\beta_1,\, \dots,\, \beta_d)\in \mathbb{Z}_+^d:=(\mathbb{N}\cup\{0\})^d$, $|\overline{\beta}|=\beta_1+\ldots+\beta_d$. For any distribution $f$ defined on $\Omega$ we write $\nabla^r f=\left(\partial^r f/\partial x^{\overline{\beta}}\right)_{|\overline{\beta}|=r}$ (here partial derivatives are taken in the sense of distributions), and denote by $l_{r,d}$ the number of components of the vector-valued distribution $\nabla^r f$. We set

$$W^r_{p,g}(\Omega)=\left\{f:\; \Omega\rightarrow \mathbb{R}\big|\; \exists\psi:\; \Omega\rightarrow \mathbb{R}^{l_{r,d}}:\; \|\psi\|_{L_p(\Omega)}\leqslant 1,\, \nabla^r f=g\cdot\psi\right\}$$

$\Big($we denote the corresponding function $\psi$ by $\dfrac{\nabla^r f}{g}\Big)$,

$$\|f\|_{L_{q,v}(\Omega)}=\|f\|_{q,v}=\|fv\|_{L_q(\Omega)}, \qquad L_{q,v}(\Omega)=\left\{f:\Omega\rightarrow \mathbb{R}|\;\; \|f\|_{q,v}<\infty\right\}.$$

We call the set $W^r_{p,g}(\Omega)$ a weighted Sobolev class. Observe that $W^r_{p,1}(\Omega)=W^r_p(\Omega)$ is a non-weighted Sobolev class. For properties of weighted Sobolev spaces and their generalizations, we refer the reader to the books [13, 14, 21, 27, 36, 37] and the survey paper [20].

Let $(X, \|\cdot\|_X)$ be a normed space, let $X^*$ be its dual, and let $\mathcal{L}_n(X)$, $n \in \mathbb{Z}_+$, be the family of subspaces of $X$ of dimension at most $n$. Denote by $L(X, Y)$ the space of continuous linear operators from $X$ into a normed space $Y$. Also, by $\operatorname{rk} A$ denote the dimension of the image of an operator $A \in L(X, Y)$, and by $\|A\|_{X\to Y}$, its norm.

By the Kolmogorov $n$-width of a set $M \subset X$ in the space $X$, we mean the quantity

$$d_n(M, X) = \inf_{L\in\mathcal{L}_n(X)} \sup_{x\in M} \inf_{y\in L} \|x-y\|_X,$$

by the linear $n$-width, the quantity

$$\lambda_n(M, X) = \inf_{A\in L(X, X),\, \mathrm{rk} A\leqslant n} \sup_{x\in M} \|x - Ax\|_X,$$

and by the Gelfand $n$-width, the quantity

$$d^n(M, X) = \inf_{x_1^*, \dots, x_n^* \in X^*} \sup\{\|x\| :\, x \in M,\, x_j^*(x) = 0,\ 1 \leqslant j \leqslant n\} =$$

$$= \inf_{A\in L(X,\mathbb{R}^n)} \sup\{\|x\| :\, x \in M \cap \ker A\}.$$

In estimating Kolmogorov, linear, and Gelfand widths we set, respectively, $\vartheta_l(M, X) = d_l(M, X)$ and $\hat{q} = q$, $\vartheta_l(M, X) = \lambda_l(M, X)$ and $\hat{q} = \min\{q, p'\}$, $\vartheta_l(M, X) = d^l(M, X)$ and $\hat{q} = p'$.

In the 1960-1980s problems concerning the values of the widths of function classes in $L_q$ and of finite-dimensional balls $B_p^n$ in $l_q^n$ were intensively studied. Here $l_q^n$ $(1 \leqslant q \leqslant \infty)$ is the space $\mathbb{R}^n$ with the norm

$$\|(x_1, \dots, x_n)\|_q \equiv \|(x_1, \dots, x_n)\|_{l_q^n} = \begin{cases} (|x_1|^q + \dots + |x_n|^q)^{1/q}, & \text{if } q < \infty, \\ \max\{|x_1|, \dots, |x_n|\}, & \text{if } q = \infty, \end{cases}$$

$B_p^n$ is the unit ball in $l_p^n$. For more details, see [29, 33, 34].

Let us formulate the result on widths of non-weighted Sobolev classes on a cube in the space $L_q$. We set

$$\theta_{p,q,r,d} = \begin{cases} \frac{\delta}{d} - \left(\frac{1}{q} - \frac{1}{p}\right)_+, & \text{if} \quad p \geqslant q \quad \text{or} \quad \hat{q} \leqslant 2, \\ \min\left\{\frac{\delta}{d} + \min\left\{\frac{1}{p} - \frac{1}{q}, \frac{1}{2} - \frac{1}{\hat{q}}\right\}, \frac{\hat{q}\delta}{2d}\right\}, & \text{if} \quad p < q,\ \hat{q} > 2. \end{cases} \tag{1}$$

**Theorem A.** (see, e.g., [7, 12, 19, 35]). *Let $r \in \mathbb{N}$, $1 \leqslant p, q \leqslant \infty$, $\frac{r}{d} + \frac{1}{q} - \frac{1}{p} > 0$. In addition, we suppose that*

$$\frac{\delta}{d} + \min\left\{\frac{1}{p} - \frac{1}{q}, \frac{1}{2} - \frac{1}{\hat{q}}\right\} \neq \frac{\hat{q}\delta}{2d} \tag{2}$$

*in the case $p < q$ and $\hat{q} > 2$. Then*

$$\vartheta_n(W_p^r([0, 1]^d), L_q([0, 1]^d)) \underset{r,d,p,q}{\asymp} n^{-\theta_{p,q,r,d}}.$$

The problem concerning estimates of widths of weighted Sobolev classes in weighted $L_q$-space was studied by Birman and Solomyak [7], El Kolli [15], Triebel [36, 38], Mynbaev and Otelbaev [27], Boykov [8, 9], Lizorkin and Otelbaev [26, 28], Aitenova and Kusainova [1, 2]. For details, see, e.g., [46].

Let $|\cdot|$ be a norm on $\mathbb{R}^d$, and let $E$, $E' \subset \mathbb{R}^d$, $x \in \mathbb{R}^d$. We set

$$\operatorname{diam}_{|\cdot|} E = \sup\{|y-z| : \, y, \, z \in E\}, \quad \operatorname{dist}_{|\cdot|}(x, \, E) = \inf\{|x-y| : \, y \in E\}.$$

**Definition 1.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, and let $a > 0$. We say that $\Omega \in \mathbf{FC}(a)$ if there exists a point $x_* \in \Omega$ such that, for any $x \in \Omega$, there exist a number $T(x) > 0$ and a curve $\gamma_x : [0, \, T(x)] \to \Omega$ with the following properties:*

1. *$\gamma_x \in AC[0, \, T(x)]$, $\left|\frac{d\gamma_x(t)}{dt}\right| = 1$ a.e.,*
2. *$\gamma_x(0) = x$, $\gamma_x(T(x)) = x_*$,*
3. *$B_{at}(\gamma_x(t)) \subset \Omega$ for any $t \in [0, \, T(x)]$.*

**Definition 2.** *We say that $\Omega$ satisfies the John condition (and call $\Omega$ a John domain) if $\Omega \in \mathbf{FC}(a)$ for some $a > 0$.*

For a bounded domain the John condition coincides with the flexible cone condition (see definition in [6]). Reshetnyak [30,31] found an integral representation for functions on a John domain $\Omega$ in terms of their derivatives of order $r$. This representation yields that for $\frac{r}{d} - \left(\frac{1}{p} - \frac{1}{q}\right)_+ \geqslant 0$ (for $\frac{r}{d} - \left(\frac{1}{p} - \frac{1}{q}\right)_+ > 0$, respectively) the class $W^r_p(\Omega)$ can be continuously (respectively, compactly) imbedded into $L_q(\Omega)$ (i.e., the conditions of continuous and compact imbeddings are the same as for $\Omega = [0, \, 1]^d$). Moreover, in [5, 39] it was proved that if $\Omega$ is a John domain and $p$, $q$, $r$, $d$ are such as in Theorem A, then widths have the same orders as for $\Omega = [0, \, 1]^d$.

Throughout we suppose that $\overline{\Omega} \subset \left(-\frac{1}{2}, \, \frac{1}{2}\right)^d$ (here $\overline{\Omega}$ is the closure of $\Omega$).

Denote by $\mathbb{H}$ the set of all non-decreasing functions defined on $(0, \, 1]$.

**Definition 3.** *(see [10]). Let $\Gamma \subset \mathbb{R}^d$ be a nonempty compact set and $h \in \mathbb{H}$. We say that $\Gamma$ is an $h$-set if there are a constant $c_* \geqslant 1$ and a finite countably additive measure $\mu$ on $\mathbb{R}^d$ such that* $\operatorname{supp} \mu = \Gamma$ *and*

$$c_*^{-1} h(t) \leqslant \mu(B_t(x)) \leqslant c_* h(t) \tag{3}$$

*for any $x \in \Gamma$ and $t \in (0, \, 1]$.*

Throughout we suppose that $1 < p \leqslant \infty$, $1 \leqslant q < \infty$, $r \in \mathbb{N}$, $\delta := r + \frac{d}{q} - \frac{d}{p} > 0$. We denote $\log x := \log_2 x$.

Let $\Gamma \subset \partial\Omega$ be an $h$-set,

$$g(x) = \varphi_g(\operatorname{dist}_{|\cdot|}(x, \, \Gamma)), \quad v(x) = \varphi_v(\operatorname{dist}_{|\cdot|}(x, \, \Gamma)), \tag{4}$$

where $\varphi_g, \varphi_v : (0, \infty) \to (0, \infty)$. Suppose that in some neighborhood of zero

$$h(t) = t^\theta |\log t|^\gamma \tau(|\log t|), \quad 0 < \theta < d, \tag{5}$$

$$\varphi_g(t) = t^{-\beta_g} |\log t|^{-\alpha_g} \rho_g(|\log t|), \quad \varphi_v(t) = t^{-\beta_v} |\log t|^{-\alpha_v} \rho_v(|\log t|), \tag{6}$$

where $\rho_g, \rho_v, \tau$ are absolutely continuous functions,

$$\lim_{y\to+\infty} \frac{y\tau'(y)}{\tau(y)} = \lim_{y\to+\infty} \frac{y\rho_g'(y)}{\rho_g(y)} = \lim_{y\to+\infty} \frac{y\rho_v'(y)}{\rho_v(y)} = 0. \tag{7}$$

For $\beta_v < \frac{d-\theta}{q}$, in [40,41,43] there were obtained sufficient conditions for embedding of $W^r_{p,g}(\Omega)$ into $L_{q,v}(\Omega)$, and order estimates of Kolmogorov, Gelfand and linear widths were found. Here we consider the limiting case

$$\beta_v = \frac{d-\theta}{q}, \quad \alpha_v > \frac{1-\gamma}{q}. \tag{8}$$

We set $\beta = \beta_g + \beta_v$, $\alpha = \alpha_g + \alpha_v$, $\rho(y) = \rho_g(y)\rho_v(y)$, $\mathfrak{Z} = (p, q, r, d, a, c_*, h, g, v)$, $\mathfrak{Z}_* = (\mathfrak{Z}, \operatorname{diam}\Omega)$.

**Theorem 1.** *There exists $n_0 = n_0(\mathfrak{Z})$ such that for any $n \geqslant n_0$ the following assertion holds.*

*1. Let $\beta - \delta + \theta \left(\frac{1}{q} - \frac{1}{p}\right)_+ < 0$. We set*

$$\sigma_*(n) = (\log n)^{-\alpha + \frac{1}{q} + \frac{(\beta-\delta)\gamma}{\theta}} \rho(\log n) \tau^{\frac{\beta-\delta}{\theta}}(\log n).$$

- *Let $p \geqslant q$ or $p < q$, $\hat{q} \leqslant 2$. We set*

$$\theta_1 = \frac{\delta}{d} - \left(\frac{1}{q} - \frac{1}{p}\right)_+, \quad \theta_2 = \frac{\delta - \beta}{\theta} - \left(\frac{1}{q} - \frac{1}{p}\right)_+, \tag{9}$$

$$\sigma_1(n) = 1, \quad \sigma_2(n) = \sigma_*(n). \tag{10}$$

*Suppose that $\theta_1 \neq \theta_2$, $j_* \in \{1, 2\}$,*

$$\theta_{j_*} = \min\{\theta_1, \theta_2\}.$$

*Then*

$$\vartheta_n(W^r_{p,g}(\Omega), L_{q,v}(\Omega)) \underset{\mathfrak{Z}_*}{\asymp} n^{-\theta_{j_*}} \sigma_{j_*}(n).$$

- *Let $p < q$, $\hat{q} > 2$. We set*

$$\theta_1 = \frac{\delta}{d} + \min\left\{\frac{1}{p} - \frac{1}{q}, \frac{1}{2} - \frac{1}{\hat{q}}\right\}, \quad \theta_2 = \frac{\hat{q}\delta}{2d}, \tag{11}$$

$$\theta_3 = \frac{\delta - \beta}{\theta} + \min\left\{\frac{1}{p} - \frac{1}{q}, \frac{1}{2} - \frac{1}{\hat{q}}\right\}, \quad \theta_4 = \frac{\hat{q}(\delta - \beta)}{2\theta}, \tag{12}$$

$$\sigma_1(n) = \sigma_2(n) = 1, \quad \sigma_3(n) = \sigma_4(n) = \sigma_*(n). \tag{13}$$

*Suppose that there exists $j_* \in \{1, 2, 3, 4\}$ such that*

$$\theta_{j_*} < \min_{j \neq j_*} \theta_j. \tag{14}$$

*Then*

$$\vartheta_n(W^r_{p,g}(\Omega), L_{q,v}(\Omega)) \underset{\mathfrak{Z}_*}{\asymp} n^{-\theta_{j_*}} \sigma_{j_*}(n).$$

2. *Let $\beta - \delta + \theta \left(\frac{1}{q} - \frac{1}{p}\right)_+ = 0$. In addition, we suppose that $\alpha_0 := \alpha - \frac{1}{q} > 0$ for $p < q$ and $\alpha_0 := \alpha - 1 - (1 - \gamma)\left(\frac{1}{q} - \frac{1}{p}\right) > 0$ for $p \geqslant q$. Then*

$$\vartheta_n(W^r_{p,g}(\Omega), L_{q,v}(\Omega)) \underset{\mathfrak{Z}_*}{\asymp} (\log n)^{-\alpha_0} \rho(\log n) \tau^{-\left(\frac{1}{q} - \frac{1}{p}\right)_+}(\log n).$$

**Remark 1.** *From Theorem A it follows that for $\frac{\delta - \beta}{\theta} > \frac{\delta}{d}$ the order estimates are the same as in the non-weighted case.*

**Remark 2.** *Formulas in Theorem 1 differ from formulas in [43] by the power of the logarithmic factor.*

The upper estimates follow from the general result about the estimate of widths of function classes on sets with tree-like structure. Problems on estimating widths and entropy numbers for embedding operators of weighted function classes on trees were studied in papers of Evans, Harris, Lang, Solomyak, Lifshits and Linde [16, 23–25, 32].

Without loss of generality, as $|\cdot|$ we may take $|(x_1, \ldots, x_d)| = \max_{1 \leqslant i \leqslant d} |x_i|$.

# 2 Proof of the upper estimate

In this section, we obtain upper estimates for widths in Theorem 1.

The following lemma was proved in [44] (see inequalities (60)).

**Lemma 1.** *Let $\Lambda_* : (0, \infty) \to (0, \infty)$ be an absolutely continuous function such that $\lim\limits_{y\to+\infty} \frac{y\Lambda_*'(y)}{\Lambda_*(y)} = 0$. Then for any $\varepsilon > 0$*

$$t^{-\varepsilon} \underset{\varepsilon,\Lambda_*}{\lesssim} \frac{\Lambda_*(ty)}{\Lambda_*(y)} \underset{\varepsilon,\Lambda_*}{\lesssim} t^{\varepsilon}, \quad 1 \leqslant y < \infty, \ 1 \leqslant t < \infty. \tag{15}$$

Let $c_* \geqslant 1$ be the constant from the definition of an $h$-set. From (5), (6), (7) and Lemma 1 it follows that there exists $c_0 = c_0(\mathfrak{Z}) \geqslant c_*$ such that

$$\frac{h(t)}{h(s)} \leqslant c_0, \quad \frac{\varphi_g(t)}{\varphi_g(s)} \leqslant c_0, \quad \frac{\varphi_v(t)}{\varphi_v(s)} \leqslant c_0, \quad j \in \mathbb{N}, \ t, \ s \in [2^{-j-1}, 2^{-j+1}]. \tag{16}$$

Let $(\Omega, \Sigma, \nu)$ be a measure space. We say that sets $A$, $B \subset \Omega$ do not overlap if $\nu(A \cap B) = 0$. Let $m \in \mathbb{N} \cup \{\infty\}$, $E$, $E_1, \dots, E_m \subset \Omega$ be measurable sets. We say that $\{E_i\}_{i=1}^m$ is a partition of $E$ if the sets $E_i$ do not overlap pairwise and $\nu\left[(\cup_{i=1}^m E_i) \triangle E\right] = 0$.

Let $(\mathcal{T}, \xi_0)$ be a tree rooted at $\xi_0$. We introduce a partial order on $\mathbf{V}(\mathcal{T})$ as follows: we say that $\xi' > \xi$ if there exists a simple path $(\xi_0, \xi_1, \dots, \xi_n, \xi')$ such that $\xi = \xi_k$ for some $k \in \overline{0, n}$. In this case, we set $\rho_{\mathcal{T}}(\xi, \xi') = \rho_{\mathcal{T}}(\xi', \xi) = n + 1 - k$. In addition, we denote $\rho_{\mathcal{T}}(\xi, \xi) = 0$. If $\xi' > \xi$ or $\xi' = \xi$, then we write $\xi' \geqslant \xi$ and denote $[\xi, \xi'] := \{\xi'' \in \mathbf{V}(\mathcal{T}) : \xi \leqslant \xi'' \leqslant \xi'\}$. This partial order on $\mathcal{T}$ induces a partial order on its subtree.

Given $j \in \mathbb{Z}_+$, $\xi \in \mathbf{V}(\mathcal{T})$, we set

$$\mathbf{V}_j(\xi) := \mathbf{V}_j^{\mathcal{T}}(\xi) := \{\xi' \geqslant \xi : \ \rho_{\mathcal{T}}(\xi, \xi') = j\}.$$

For each vertex $\xi \in \mathbf{V}(\mathcal{T})$ we denote by $\mathcal{T}_\xi = (\mathcal{T}_\xi, \xi)$ a subtree in $\mathcal{T}$ with vertex set $\{\xi' \in \mathbf{V}(\mathcal{T}) : \xi' \geqslant \xi\}$.

In [40, 41] a tree $(\mathcal{A}, \eta_{j_*,1})$ with vertex set $\{\eta_{j,i}\}_{j\geqslant j_*, \, i\in \tilde{I}_j}$ was constructed, as well as the partition of $\Omega$ into subdomains $\Omega[\xi]$, $\xi \in \mathbf{V}(\mathcal{A})$. Moreover, $\mathbf{V}^{\mathcal{A}}_{j-j_*}(\eta_{j_*,1}) = \{\eta_{j,i}\}_{i\in\tilde{I}_j}$ and there exists a number $\overline{s} = \overline{s}(a, d) \in \mathbb{N}$ such that

$$\operatorname{diam}\Omega[\eta_{j,i}] \underset{a,d,c_0}{\asymp} 2^{-\overline{s}j}, \quad \operatorname{dist}_{|\cdot|}(x, \Gamma) \underset{a,d,c_0}{\asymp} 2^{-\overline{s}j}, \quad x \in \Omega[\eta_{j,i}],$$

$$\operatorname{card} \mathbf{V}^{\mathcal{A}}_{j'-j}(\eta_{j,i}) \underset{a,d,c_0}{\lesssim} \frac{h(2^{-\overline{s}j})}{h(2^{-\overline{s}j'})}, \quad j' \geqslant j \geqslant j_*.$$

In particular,

$$\operatorname{card} \mathbf{V}^{\mathcal{A}}_1(\eta_{j,i}) \overset{(16)}{\underset{a,d,c_0}{\lesssim}} 1, \quad j \geqslant j_*. \tag{17}$$

We set

$$u(\eta_{j,i}) = u_j = \varphi_g(2^{-\overline{s}j}) \cdot 2^{-\left(r-\frac{d}{p}\right)\overline{s}j}, \quad w(\eta_{j,i}) = w_j = \varphi_v(2^{-\overline{s}j}) \cdot 2^{-\frac{d\overline{s}j}{q}}. \tag{18}$$

Given a subtree $\mathcal{D} \subset \mathcal{A}$, we denote $\Omega[\mathcal{D}] = \cup_{\xi\in\mathbf{V}(\mathcal{D})}\Omega[\xi]$.

In [45] sufficient conditions for embedding $W^r_{p,g}(\Omega)$ into $L_{q,v}(\Omega)$ were obtained; here (16) holds and the functions $g$, $v$ satisfy (4). Let us formulate these results.

**Theorem B.** *Let $u$, $w$ be defined by (18), $1 < p < q < \infty$. Suppose that there exist $l_0 \in \mathbb{N}$ and $\lambda \in (0, 1)$ such that*

$$\frac{\left(\sum\limits_{i=j+l_0}^{\infty} \frac{h(2^{-\overline{s}(j+l_0)})}{h(2^{-\overline{s}i})} w_i^q\right)^{1/q}}{w_j} \leqslant \lambda, \quad j \geqslant j_*. \tag{19}$$

*Let $\sup_{j\geqslant j_*} u_j \left(\sum\limits_{i=j}^{\infty} \frac{h(2^{-\overline{s}j})}{h(2^{-\overline{s}i})} w_i^q\right)^{1/q} < \infty$. Then $W^r_{p,g}(\Omega) \subset L_{q,v}(\Omega)$ and for any $k \geqslant j_*$, $\xi_* \in \mathbf{V}^{\mathcal{A}}_{k-j_*}(\eta_{j_*,1})$ there exists a linear continuous operator $P : L_{q,v}(\Omega) \to \mathcal{P}_{r-1}(\Omega)$ such that for any subtree $\mathcal{D} \subset \mathcal{A}$ rooted at $\xi_*$ and for any function $f \in W^r_{p,g}(\Omega)$*

$$\|f - Pf\|_{L_{q,v}(\Omega[\mathcal{D}])} \underset{\mathfrak{Z}}{\lesssim} \sup_{j\geqslant k} u_j \left(\sum_{i\geqslant j} \frac{h(2^{-\overline{s}j})}{h(2^{-\overline{s}i})} w_i^q\right)^{\frac{1}{q}} \left\|\frac{\nabla^r f}{g}\right\|_{L_p(\Omega[\mathcal{D}])}.$$

**Theorem C.** *Let $p \geqslant q$, $\xi_* \in \mathbf{V}^{\mathcal{A}}_{k-j_*}(\eta_{j_*,1})$, and let the functions $u$, $w$ on $\mathbf{V}(\mathcal{A})$ be defined by (18). We set $\hat{w}_j = w_j \cdot \left(\frac{h(2^{-\overline{s}k})}{h(2^{-\overline{s}j})}\right)^{\frac{1}{q}}$, $\hat{u}_j = u_j \cdot \left(\frac{h(2^{-\overline{s}j})}{h(2^{-\overline{s}k})}\right)^{\frac{1}{p}}$, $k \leqslant j < \infty$. Let*

$$M_{\hat{u},\hat{w}}(k) := \sup_{k\leqslant j<\infty} \Big(\sum_{i=j}^{\infty} \hat{w}_i^q\Big)^{\frac{1}{q}} \Big(\sum_{i=k}^{j} \hat{u}_i^{p'}\Big)^{\frac{1}{p'}} < \infty, \quad 1 < p = q < \infty, \tag{20}$$

$$M_{\hat{u},\hat{w}}(k) := \left(\sum_{j=k}^{\infty} \left(\Big(\sum_{i=j}^{\infty} \hat{w}_i^q\Big)^{\frac{1}{p}} \Big(\sum_{i=k}^{j} \hat{u}_i^{p'}\Big)^{\frac{1}{p'}}\right)^{\frac{pq}{p-q}} \hat{w}_j^q\right)^{\frac{1}{q}-\frac{1}{p}} < \infty, \quad q < p. \tag{21}$$

*Then $W^r_{p,g}(\Omega[\mathcal{A}_{\xi_*}]) \subset L_{q,v}(\Omega[\mathcal{A}_{\xi_*}])$ and there exists a linear continuous operator $P : L_{q,v}(\Omega) \to \mathcal{P}_{r-1}(\Omega)$ such that for any subtree $\mathcal{D} \subset \mathcal{A}_{\xi_*}$ rooted at $\xi_*$ and for any function $f \in W^r_{p,g}(\Omega)$*

$$\|f - Pf\|_{L_{q,v}(\Omega[\mathcal{D}])} \underset{\mathfrak{Z}}{\lesssim} M_{\hat{u},\hat{w}}(k) \left\|\frac{\nabla^r f}{g}\right\|_{L_p(\Omega[\mathcal{D}])}.$$

Suppose that (5), (6), (7), (8) hold.

From (6), (8) and (18) it follows that

$$u(\eta_{j,i}) = u_j = 2^{\overline{s}j\left(\beta_g - r + \frac{d}{p}\right)} (\overline{s}j)^{-\alpha_g} \rho_g(\overline{s}j), \quad w(\eta_{j,i}) = w_j = 2^{-\frac{\theta\overline{s}j}{q}} (\overline{s}j)^{-\alpha_v} \rho_v(\overline{s}j). \tag{22}$$

Recall that $\delta = r + \frac{d}{q} - \frac{d}{p}$.

**Corollary 1.** *Let $1 < p < q < \infty$, $r \in \mathbb{N}$, $\delta > 0$, and let the conditions (5), (6), (7), (8) hold. In addition, we suppose that*

$$\text{either} \quad \beta - \delta < 0 \quad \text{or} \quad \beta - \delta = 0, \quad \alpha > \frac{1}{q}. \tag{23}$$

*Then $W^r_{p,g}(\Omega) \subset L_{q,v}(\Omega)$ and for any $k \geqslant j_*$, $\xi_* \in \mathbf{V}^{\mathcal{A}}_{k-j_*}(\eta_{j_*,1})$ there exists a linear continuous operator $P : L_{q,v}(\Omega) \to \mathcal{P}_{r-1}(\Omega)$ such that for any subtree $\mathcal{D} \subset \mathcal{A}$ rooted at $\xi_*$ and for any function $f \in W^r_{p,g}(\Omega)$*

$$\|f - Pf\|_{L_{q,v}(\Omega[\mathcal{D}])} \underset{\mathfrak{Z}}{\lesssim} 2^{-(\delta-\beta)\overline{s}k} (\overline{s}k)^{-\alpha+\frac{1}{q}} \rho(\overline{s}k) \left\| \frac{\nabla^r f}{g} \right\|_{L_p(\Omega[\mathcal{D}])}.$$

**Proof.** From (5) and (22) it follows that

$$\begin{aligned}\sum_{i=j}^{\infty} \tfrac{h(2^{-\overline{s}j})}{h(2^{-\overline{s}i})} w_i^q = \sum_{i=j}^{\infty} 2^{-\theta\overline{s}i} (\overline{s}i)^{-\alpha_v q} \rho_v^q(\overline{s}i) \cdot \tfrac{2^{\theta\overline{s}i} (\overline{s}j)^{\gamma} \tau(\overline{s}j)}{2^{\theta\overline{s}j} (\overline{s}i)^{\gamma} \tau(\overline{s}i)} \underset{\mathfrak{Z}}{\overset{(8),(15)}{\asymp}} \\ \asymp 2^{-\theta\overline{s}j} [\overline{s}j]^{-\alpha_v q+1} \rho_v^q(\overline{s}j).\end{aligned} \tag{24}$$

This together with Lemma 1 implies (19). Further,

$$\sup_{j \geqslant k} u_j \left( \sum_{i \geqslant j} \frac{h(2^{-\overline{s}j})}{h(2^{-\overline{s}i})} w_i^q \right)^{\frac{1}{q}} \underset{\mathfrak{Z}}{\overset{(8),(22),(23),(24)}{\asymp}} 2^{-(\delta-\beta)\overline{s}k} (\overline{s}k)^{-\alpha+\frac{1}{q}} \rho(\overline{s}k).$$

It remains to apply Theorem B. □

Let us consider the case $p \geqslant q$. We apply Theorem C. For $j \geqslant k$, we have

$$\begin{aligned}\hat{u}_j \overset{(5),(22)}{=} 2^{\overline{s}j\left(\beta_g - r + \frac{d}{p}\right)} (\overline{s}j)^{-\alpha_g} \rho_g(\overline{s}j) \cdot 2^{-\frac{\theta\overline{s}(j-k)}{p}} \frac{j^{\frac{\gamma}{p}} \tau^{\frac{1}{p}}(\overline{s}j)}{k^{\frac{\gamma}{p}} \tau^{\frac{1}{p}}(\overline{s}k)}, \\ \hat{w}_j \overset{(5),(22)}{=} 2^{-\frac{\theta\overline{s}k}{q}} (\overline{s}j)^{-\alpha_v} \rho_v(\overline{s}j) \cdot \frac{k^{\frac{\gamma}{q}} \tau^{\frac{1}{q}}(\overline{s}k)}{j^{\frac{\gamma}{q}} \tau^{\frac{1}{q}}(\overline{s}j)}.\end{aligned} \tag{25}$$

**Corollary 2.** *Let $1 < p \leqslant \infty$, $1 \leqslant q < \infty$, $p \geqslant q$, $r \in \mathbb{N}$, $\delta > 0$ and let conditions (5), (6), (7), (8) hold. Suppose that either $\beta - \delta + \theta\left(\frac{1}{q} - \frac{1}{p}\right) < 0$ or $\beta - \delta + \theta\left(\frac{1}{q} - \frac{1}{p}\right) = 0$ and $\alpha - 1 - (1-\gamma)\left(\frac{1}{q} - \frac{1}{p}\right) > 0$. Then $W^r_{p,g}(\Omega) \subset L_{q,v}(\Omega)$ and for any $k \geqslant j_*$, $\xi_* \in \mathbf{V}^{\mathcal{A}}_{k-j_*}(\eta_{j_*,1})$ there exists a linear continuous operator $P : L_{q,v}(\Omega) \to \mathcal{P}_{r-1}(\Omega)$ such that for any subtree $\mathcal{D} \subset \mathcal{A}$ rooted at $\xi_*$ and for any function $f \in W^r_{p,g}(\Omega)$*

$$\|f - Pf\|_{L_{q,v}(\Omega[\mathcal{D}])} \underset{\mathfrak{Z}}{\lesssim} 2^{-(\delta-\beta)\overline{s}k} (\overline{s}k)^{-\alpha+\frac{1}{q}} \rho(\overline{s}k) \left\| \frac{\nabla^r f}{g} \right\|_{L_p(\Omega[\mathcal{D}])}$$

*in the case $\beta - \delta + \theta\left(\frac{1}{q} - \frac{1}{p}\right) < 0$, and*

$$\|f - Pf\|_{L_{q,v}(\Omega[\mathcal{D}])} \underset{\mathfrak{Z}}{\lesssim} 2^{-\theta\left(\frac{1}{q} - \frac{1}{p}\right)\overline{s}k} (\overline{s}k)^{-\alpha+1+\frac{1}{q}-\frac{1}{p}} \rho(\overline{s}k) \left\| \frac{\nabla^r f}{g} \right\|_{L_p(\Omega[\mathcal{D}])}$$

*in the case $\beta - \delta + \theta\left(\frac{1}{q} - \frac{1}{p}\right) = 0$.*

**Proof.** Let $p = q$. Applying (25) and (20) and taking into account that $\alpha_v > \frac{1-\gamma}{q}$ and $\beta_g - r + \frac{d}{p} - \frac{\theta}{p} \overset{(8)}{=} \beta - \delta$, we get

$$M_{\hat{u},\hat{w}}(k) \underset{\mathfrak{Z}}{\lesssim} \sup_{l\geqslant k}(\overline{s}l)^{-\alpha_v+\frac{1-\gamma}{q}}\rho_v(\overline{s}l)\tau^{-\frac{1}{q}}(\overline{s}l)\left(\sum_{j=k}^{l} 2^{p'(\beta-\delta)\overline{s}j}(\overline{s}j)^{p'\left(-\alpha_g+\frac{\gamma}{p}\right)}\rho_g^{p'}(\overline{s}j)\tau^{\frac{p'}{p}}(\overline{s}j)\right)^{\frac{1}{p'}}.$$

If $\beta - \delta < 0$, then by Lemma 1

$$M_{\hat{u},\hat{w}}(k) \underset{\mathfrak{Z}}{\lesssim} 2^{(\beta-\delta)\overline{s}k}(\overline{s}k)^{-\alpha+\frac{1}{q}}\rho(\overline{s}k). \tag{26}$$

Let $\beta - \delta = 0$. We may assume that $-\alpha_g + \frac{\gamma}{p} + \frac{1}{p'} > 0$ (otherwise, we multiply $\hat{u}_j$ by $\frac{j^c}{k^c}$ with some $c > 0$). Then

$$M_{\hat{u},\hat{w}}(k) \underset{\mathfrak{Z}}{\lesssim} (\overline{s}k)^{-\alpha+1}\rho(\overline{s}k). \tag{27}$$

Let $p > q$. Applying (25) and (21) and taking into account that $\alpha_v > \frac{1-\gamma}{q}$ and $\beta_g - r + \frac{d}{p} - \frac{\theta}{p} \overset{(8)}{=} \beta - \delta + \theta\left(\frac{1}{q} - \frac{1}{p}\right)$, we get

$$M_{\hat{u},\hat{w}}(k) \underset{\mathfrak{Z}}{\overset{(8)}{\lesssim}} 2^{-\theta\overline{s}k\left(\frac{1}{q}-\frac{1}{p}\right)}(\overline{s}k)^{\gamma\left(\frac{1}{q}-\frac{1}{p}\right)}\tau^{\frac{1}{q}-\frac{1}{p}}(\overline{s}k)\times$$

$$\times\left(\sum_{j=k}^{\infty}(\overline{s}j)^{\frac{pq}{p-q}\left(-\alpha_v-\frac{\gamma}{q}+\frac{1}{p}\right)}\rho_v^{\frac{pq}{p-q}}(\overline{s}j)\tau^{-\frac{p}{p-q}}(\overline{s}j)\sigma(j)^{\frac{pq}{p-q}}\right)^{\frac{1}{q}-\frac{1}{p}},$$

where

$$\sigma(j) = \left(\sum_{i=k}^{j} 2^{\overline{s}i\left(\beta-\delta+\theta\left(\frac{1}{q}-\frac{1}{p}\right)\right)p'}(\overline{s}i)^{p'\left(-\alpha_g+\frac{\gamma}{p}\right)}\rho_g^{p'}(\overline{s}i)\tau^{\frac{p'}{p}}(\overline{s}i)\right)^{\frac{1}{p'}}.$$

If $\beta - \delta + \theta\left(\frac{1}{q} - \frac{1}{p}\right) < 0$, then

$$\sigma(j) \underset{\mathfrak{Z}}{\lesssim} 2^{\overline{s}k\left(\beta-\delta+\theta\left(\frac{1}{q}-\frac{1}{p}\right)\right)}(\overline{s}k)^{\left(-\alpha_g+\frac{\gamma}{p}\right)}\rho_g(\overline{s}k)\tau^{\frac{1}{p}}(\overline{s}k),$$

and by the second relation in (8) we have

$$M_{\hat{u},\hat{w}}(k) \underset{\mathfrak{Z}}{\lesssim} 2^{(\beta-\delta)\overline{s}k}(\overline{s}k)^{-\alpha+\frac{1}{q}}\rho(\overline{s}k). \tag{28}$$

If $\beta - \delta + \theta\left(\frac{1}{q} - \frac{1}{p}\right) = 0$ and $\alpha > 1 + (1-\gamma)\left(\frac{1}{q} - \frac{1}{p}\right)$, then we may assume that $-\alpha_g + \frac{\gamma}{p} + \frac{1}{p'} > 0$. We have

$$M_{\hat{u},\hat{w}}(k) \underset{\mathfrak{Z}}{\lesssim} 2^{-\theta\left(\frac{1}{q}-\frac{1}{p}\right)\overline{s}k}(\overline{s}k)^{-\alpha+1+\frac{1}{q}-\frac{1}{p}}\rho(\overline{s}k). \tag{29}$$

This completes the proof. $\square$

**Remark 3.** *Notice that in order to prove Theorems B and C we use estimates for norms of summation operators on trees, which are obtained in [42]. If* $\beta-\delta+\theta\left(\frac{1}{q}-\frac{1}{p}\right)_+<0$*, then these estimates can be proved easier (we argue similarly as in [41, Lemma 5.1]).*

Applying Corollaries 1 and 2 and arguing similarly as in [43, Theorem 1], we obtain the desired upper estimate of widths.

# 3 Proof of the lower estimate

In this section, we obtain the lower estimates of widths in Theorem 1.

If $\frac{\delta-\beta}{\theta}>\frac{\delta}{d}$, then by Theorem A (see also Remark 1) and by the upper estimate of $\vartheta_n(W^r_{p,g}(\Omega),\, L_{q,v}(\Omega))$, which is already obtained, we have $\vartheta_n(W^r_{p,g}(\Omega),\, L_{q,v}(\Omega)) \underset{\mathfrak{Z}}{\lesssim} \vartheta_n(W^r_p([0,\,1]^d),\, L_q([0,\,1]^d))$. On the other hand, there is a cube $\Delta\subset\Omega$ with side length $l(\Delta)\underset{\mathfrak{Z}_*}{\asymp} 1$ such that $g(x)\underset{\mathfrak{Z}_*}{\asymp} 1$, $v(x)\underset{\mathfrak{Z}_*}{\asymp} 1$ for any $x\in\Delta$ (see [43]). Hence, $\vartheta_n(W^r_{p,g}(\Omega),\, L_{q,v}(\Omega)) \underset{\mathfrak{Z}_*}{\gtrsim} \vartheta_n(W^r_p([0,\,1]^d),\, L_q([0,\,1]^d))$. Thus, we obtained the order estimates of widths in the case $\frac{\delta-\beta}{\theta}>\frac{\delta}{d}$.

Consider the case $\frac{\delta-\beta}{\theta}\leqslant\frac{\delta}{d}$. In order to obtain the lower estimates we argue similarly as in [43]. It is sufficient to prove the following assertions.

**Proposition 1.** *Let* $\frac{\delta-\beta}{\theta}\leqslant\frac{\delta}{d}$*. Suppose that one of the following conditions holds: 1)* $\beta-\delta+\theta\left(\frac{1}{q}-\frac{1}{p}\right)_+<0$ *or 2)* $\beta=\delta$, $p<q$*. Then there exist* $t_0=t_0(\mathfrak{Z}_*)\in\mathbb{N}$ *and* $\hat{k}=\hat{k}(\mathfrak{Z}_*)\in\mathbb{N}$ *such that for any* $t\in\mathbb{N}$, $t\geqslant t_0$ *there exist functions* $\psi_{j,t}\in C_0^\infty(\mathbb{R}^d)$ $(1\leqslant j\leqslant j_t)$ *with pairwise non-overlapping supports such that*

$$j_t \underset{\mathfrak{Z}_*}{\gtrsim} 2^{\theta\hat{k}t}(\hat{k}t)^{-\gamma}\tau^{-1}(\hat{k}t), \tag{30}$$

$$\left\|\frac{\nabla^r\psi_{j,t}}{g}\right\|_{L_p(\Omega)}=1, \quad \|\psi_{j,t}\|_{L_{q,v}(\Omega)} \underset{\mathfrak{Z}_*}{\gtrsim} 2^{(\beta-\delta)\hat{k}t}(\hat{k}t)^{-\alpha+\frac{1}{q}}\rho(\hat{k}t). \tag{31}$$

**Proposition 2.** *Let* $\beta-\delta+\theta\left(\frac{1}{q}-\frac{1}{p}\right)=0$, $p\geqslant q$*. Then there exist* $t_0=t_0(\mathfrak{Z}_*)\in\mathbb{N}$ *and* $\hat{k}=\hat{k}(\mathfrak{Z}_*)\in\mathbb{N}$ *such that for any* $t\in\mathbb{N}$, $t\geqslant t_0$ *there exist functions* $\psi_{j,t}\in C_0^\infty(\mathbb{R}^d)$ $(1\leqslant j\leqslant j_t)$ *with pairwise non-overlapping supports such that*

$$j_t \underset{\mathfrak{Z}_*}{\gtrsim} 2^{\theta\hat{k}t}(\hat{k}t)^{-\gamma}\tau^{-1}(\hat{k}t), \tag{32}$$

$$\left\|\frac{\nabla^r\psi_{j,t}}{g}\right\|_{L_p(\Omega)}=1, \quad \|\psi_{j,t}\|_{L_{q,v}(\Omega)} \underset{\mathfrak{Z}_*}{\gtrsim} 2^{-\theta\left(\frac{1}{q}-\frac{1}{p}\right)\hat{k}t}(\hat{k}t)^{-\alpha+\frac{1}{q}+1-\frac{1}{p}}\rho(\hat{k}t). \tag{33}$$

First we formulate the Vitali covering theorem [22, p. 408]).

**Theorem D.** *Denote by $B(x,\,t)$ the open or closed ball of radius $t$ with respect to some norm on $\mathbb{R}^d$ centered in $x$. Let $E \subset \mathbb{R}^d$ be a finite union of balls $B(x_i,\,r_i)$, $1 \leqslant i \leqslant l$. Then there exists a subset $\mathcal{I} \subset \{1,\,\dots,\,l\}$ such that the balls $\{B(x_i,\,r_i)\}_{i\in\mathcal{I}}$ are pairwise non-overlapping and $E \subset \cup_{i\in\mathcal{I}} B(x_i,\,3r_i)$.*

Let $\mathcal{K}$ be a family of closed cubes in $\mathbb{R}^d$ with axes parallel to coordinate axes. Given a cube $K \in \mathcal{K}$ and $s \in \mathbb{Z}_+$, we denote by $\Xi_s(K)$ the partition of $K$ into $2^{sd}$ closed non-overlapping cubes of the same size, and we set $\Xi(K) := \bigcup_{s\in\mathbb{Z}_+} \Xi_s(K)$.

Given a cube $\Delta \in \Xi\left(\left[-\frac{1}{2},\,\frac{1}{2}\right]^d\right)$ such that $\Delta \cap \Gamma \neq \varnothing$, we define the cubes $Q_\Delta$, $\tilde{Q}_\Delta$, $\hat{Q}_\Delta$ and the points $x_\Delta$, $\hat{x}_\Delta$ as follows.

Let $m \in \mathbb{N}$, $\Delta \in \Xi_m\left(\left[-\frac{1}{2},\,\frac{1}{2}\right]^d\right)$, $\Delta \cap \Gamma \neq \varnothing$. We choose $x_\Delta \in \Delta \cap \Gamma$ and a cube $Q_\Delta$ such that $\Delta \in \Xi_1(Q_\Delta)$,

$$\operatorname{dist}_{|\cdot|}(x_\Delta,\,\partial Q_\Delta) \geqslant 2^{-m-1}. \tag{34}$$

Denote by $\hat{x}_\Delta$ the center of $Q_\Delta$. Then

$$Q_\Delta = \hat{x}_\Delta + 2^{-m+1} \cdot \left[-\frac{1}{2},\,\frac{1}{2}\right]^d. \tag{35}$$

We set

$$\tilde{Q}_\Delta = \hat{x}_\Delta + 3 \cdot 2^{-m} \cdot \left[-\frac{1}{2},\,\frac{1}{2}\right]^d, \quad \hat{Q}_\Delta = \hat{x}_\Delta + 2^{-m+2} \cdot \left[-\frac{1}{2},\,\frac{1}{2}\right]^d. \tag{36}$$

Recall that the norm $|\cdot|$ is defined by $|(x_1,\,\dots,\,x_d)| = \max_{1\leqslant i\leqslant d} |x_i|$.

Let $\hat{k} \in \mathbb{N}$ (it will be chosen later). For each $l \in \mathbb{Z}_+$ we set

$$\hat{E}_l(\Delta) = \{x \in \hat{Q}_\Delta :\ \operatorname{dist}_{|\cdot|}(x,\,\Gamma) \leqslant 2^{-m-\hat{k}l+2}\}, \quad E_l(\Delta) = \hat{E}_l(\Delta) \cap Q_\Delta \cap \Omega. \tag{37}$$

Notice that

$$\hat{Q}_\Delta = \hat{E}_0(\Delta). \tag{38}$$

Denote by $\operatorname{mes} A$ the Lebesgue measure of the measurable set $A \subset \mathbb{R}^d$.

**Lemma 2.** *The following estimate holds:*

$$\operatorname{mes} \hat{E}_l(\Delta) \underset{\mathfrak{Z}_*}{\lesssim} 2^{-md-(d-\theta)\hat{k}l} \frac{m^\gamma \tau(m)}{(m+\hat{k}l)^\gamma \tau(m+\hat{k}l)}. \tag{39}$$

*In addition, there exists $m_0 = m_0(\mathfrak{Z}_*)$ such that for $m \geqslant m_0$*

$$\operatorname{mes} E_l(\Delta) \underset{\mathfrak{Z}_*}{\gtrsim} 2^{-md-(d-\theta)\hat{k}l} \frac{m^\gamma \tau(m)}{(m+\hat{k}l)^\gamma \tau(m+\hat{k}l)}. \tag{40}$$

**Proof.** Let us prove (39). Consider the covering of the set $\hat{E}_l(\Delta)$ by cubes $x+K$, $x \in \Gamma\cap \hat{Q}_\Delta$, $K = \left(-2^{-m-\hat{k}l+3},\, 2^{-m-\hat{k}l+3}\right)$. We take a finite subcovering; applying Theorem D (the balls are taken with respect to $|\cdot|$), we get a family of pairwise non-intersecting balls $\{x_i+K\}_{i=1}^N$ such that $\{x_i+3K\}_{i=1}^N$ is a covering of $\hat{E}_l(\Delta)$. Since $\cup_{i=1}^N(x_i+K)$ is contained in a ball $B$ of radius $\tilde{R} \underset{\mathfrak{Z}_*}{\asymp} 2^{-m}$, we have

$$\sum_{i=1}^N \mu(x_i+K) \leqslant \mu(B) \overset{(3),(16)}{\underset{\mathfrak{Z}_*}{\lesssim}} h(2^{-m});$$

since $x_i\in\Gamma$, we get $\mu(x_i+K) \overset{(3),(16)}{\underset{\mathfrak{Z}_*}{\asymp}} h(2^{-m-\hat{k}l})$ and $N \underset{\mathfrak{Z}_*}{\lesssim} \frac{h(2^{-m})}{h(2^{-m-\hat{k}l})}$. Finally,

$$\operatorname{mes}\hat{E}_l(\Delta) \leqslant \sum_{i=1}^N \operatorname{mes}(x_i+3K) \underset{\mathfrak{Z}_*}{\lesssim} 2^{-(m+\hat{k}l)d}\frac{h(2^{-m})}{h(2^{-m-\hat{k}l})}.$$

It remains to apply (5).

Let us prove (40). Denote by $Q^*_\Delta$ the homothetic transform of the cube $Q_\Delta$ with respect to its center with the coefficient $1-2^{-\hat{k}l-3}$. We set

$$\{\Delta_i\}_{i=1}^L = \left\{\Delta'\in \Xi_{m+\hat{k}l+3}\left([-1/2,\,1/2]^d\right):\ \Delta'\subset Q^*_\Delta,\ \Delta'\cap\Gamma\ne\varnothing\right\}.$$

It can be proved similarly as formula (4.20) in [40] that $L \underset{\mathfrak{Z}_*}{\asymp} \frac{h(2^{-m})}{h(2^{-m-\hat{k}l})}$. Since $\Delta_i\cap\Gamma\ne\varnothing$, it follows from the definition of $\Delta_i$ and $Q^*_\Delta$ that $\cup_{i=1}^L Q_{\Delta_i}\subset \hat{E}_l(\Delta)\cap Q_\Delta$. Finally, for any $j\in\{1,\,\dots,\,L\}$

$$\operatorname{card}\{i\in\overline{1,\,L}:\ \operatorname{mes}(Q_{\Delta_i}\cap Q_{\Delta_j})>0\} \underset{d}{\lesssim} 1.$$

Therefore, it is sufficient to prove that $\operatorname{mes}(Q_{\Delta_i}\cap\Omega) \underset{\mathfrak{Z}_*}{\asymp} 2^{-(m+\hat{k}l)d}$.

Let $x\in Q_{\Delta_i}\cap\Omega$, $|x-x_{\Delta_i}|\leqslant 2^{-m-\hat{k}l-5}$. This point exists since $x_{\Delta_i}\in\Gamma\subset\partial\Omega$ and (34) holds with $m+\hat{k}l+3$ instead of $m$; moreover, $\operatorname{dist}_{|\cdot|}(x,\,\partial Q_{\Delta_i}) \underset{d}{\gtrsim} 2^{-m-\hat{k}l}$. Let $x_*$ and $\gamma_x(\cdot):[0,\,T(x)]\to\Omega$ be such as in Definition 1. There exists $m_0 = m_0(\mathfrak{Z}_*)$ such that $x_*\notin Q_{\Delta_i}$ for $m\geqslant m_0$. Let $\gamma_x(t_*)\in\partial Q_{\Delta_i}$. Then $t_* \underset{d}{\gtrsim} 2^{-m-\hat{k}l}$. By Definition 1, the ball $B_{at_*}(\gamma_x(t_*))$ is contained in $\Omega$. It remains to observe that $\operatorname{mes}\left(B_{at_*}(\gamma_x(t_*))\cap Q_{\Delta_i}\right) \underset{\mathfrak{Z}_*}{\gtrsim} 2^{-(m+\hat{k}l)d}$. $\square$

**Remark 4.** *From (39) it follows that* $\operatorname{mes}(\hat{Q}_\Delta\cap\Gamma)=0$.

Suppose that $m\geqslant m_0(\mathfrak{Z}_*)$.

Choose $\hat{k}=\hat{k}(\mathfrak{Z}_*)$ such that for any $l\in\mathbb{Z}_+$

$$\operatorname{mes}\left(E_l(\Delta)\backslash\hat{E}_{l+1}(\Delta)\right) \underset{\mathfrak{Z}_*}{\asymp} 2^{-md-(d-\theta)\hat{k}l}\frac{m^\gamma\tau(m)}{(m+\hat{k}l)^\gamma\tau(m+\hat{k}l)} \tag{41}$$

(it is possible by (15), (39) and (40)).

Let $\psi \in C_0^\infty(\mathbb{R}^d)$, $\operatorname{supp}\psi \subset \left[-\frac12, \frac12\right]^d$, $\psi|_{\left[-\frac38, \frac38\right]^d} = 1$, $\psi(x) \in [0, 1]$ for any $x \in \mathbb{R}^d$. We set

$$\psi_\Delta(x) = \psi(2^{m-2}(x - \hat{x}_\Delta)). \tag{42}$$

Then

$$\operatorname{supp}\psi_\Delta \subset \hat{Q}_\Delta, \quad \psi_\Delta|_{\tilde{Q}_\Delta} = 1, \tag{43}$$

$$\left|\frac{\nabla^r\psi_\Delta(x)}{g(x)}\right| \underset{\mathfrak{Z}_*}{\overset{(4),(6),(37)}{\lesssim}} 2^{-\beta_g(m+\hat{k}l)}(m+\hat{k}l)^{\alpha_g}\rho_g^{-1}(m+\hat{k}l)\cdot 2^{rm}, \quad x \in \hat{E}_l(\Delta)\backslash\hat{E}_{l+1}(\Delta). \tag{44}$$

We set $c_\Delta = \left\|\frac{\nabla^r\psi_\Delta}{g}\right\|^{-1}_{L_p(\hat{Q}_\Delta)} > 0$.

**Lemma 3.** *The following estimates hold:*

$$c_\Delta \underset{\mathfrak{Z}_*}{\gtrsim} 2^{\left(\beta_g - r + \frac{d}{p}\right)m} m^{-\alpha_g}\rho_g(m), \quad c_\Delta\|\psi_\Delta\|_{L_{q,v}(\Omega)} \underset{\mathfrak{Z}_*}{\gtrsim} 2^{(\beta-\delta)m} m^{-\alpha+\frac1q}\rho(m). \tag{45}$$

**Proof.** We estimate the value $\left\|\frac{\nabla^r\psi_\Delta}{g}\right\|_{L_p(\hat{Q}_\Delta)}$ from above. First we notice that from the conditions $\frac{\delta-\beta}{\theta} \leqslant \frac{\delta}{d}$, $\theta < d$ and $\beta_v \overset{(8)}{=} \frac{d-\theta}{q}$ it follows that

$$\beta_g + \frac{d-\theta}{p} > 0. \tag{46}$$

Hence, by Remark 4,

$$\left\|\frac{\nabla^r\psi_\Delta}{g}\right\|^p_{L_p(\hat{Q}_\Delta)} \overset{(38)}{=} \sum_{l\in\mathbb{Z}_+} \left\|\frac{\nabla^r\psi_\Delta}{g}\right\|^p_{L_p(\hat{E}_l(\Delta)\backslash\hat{E}_{l+1}(\Delta))} \underset{\mathfrak{Z}_*}{\overset{(39),(44)}{\lesssim}}$$

$$\lesssim \sum_{l\in\mathbb{Z}_+} 2^{-p\beta_g(m+\hat{k}l)}(m+\hat{k}l)^{p\alpha_g}\rho_g^{-p}(m+\hat{k}l)\cdot 2^{prm}\cdot 2^{-dm-(d-\theta)\hat{k}l}\frac{m^\gamma\tau(m)}{(m+\hat{k}l)^\gamma\tau(m+\hat{k}l)} \underset{\mathfrak{Z}_*}{\overset{(46)}{\asymp}}$$

$$\asymp 2^{p\left(-\beta_g+r-\frac{d}{p}\right)m} m^{p\alpha_g}\rho_g^{-p}(m).$$

This implies the first inequality in (45). Let us prove the second inequality. Taking into account that $\psi_\Delta|_{Q_\Delta} \overset{(43)}{=} 1$ and $\beta_v = \frac{d-\theta}{q}$, we get

$$\|\psi_\Delta\|^q_{L_{q,v}(\Omega)} \geqslant \sum_{l\in\mathbb{Z}_+} \|v\psi_\Delta\|^q_{L_q(E_l(\Delta)\backslash\hat{E}_{l+1}(\Delta))} \underset{\mathfrak{Z}_*}{\overset{(4),(6),(37),(41)}{\gtrsim}}$$

$$\gtrsim \sum_{l\in\mathbb{Z}_+} 2^{\beta_v q(m+\hat{k}l)}(m+\hat{k}l)^{-\alpha_v q}\rho_v^q(m+\hat{k}l)\cdot 2^{-md-(d-\theta)\hat{k}l}\frac{m^\gamma\tau(m)}{(m+\hat{k}l)^\gamma\tau(m+\hat{k}l)} \overset{(8)}{\underset{\mathfrak{Z}_*}{\gtrsim}}$$

$$\gtrsim 2^{(\beta_v q-d)m}m^{-q\alpha_v+1}\rho_v^q(m).$$

It remains to apply the first inequality in (45). □

**Proof of Proposition 1.** Let

$$\{\Delta_\nu\}_{\nu\in\mathcal{N}} = \left\{\Delta\in\Xi_{\hat{k}t}\left(\left[-\frac{1}{2},\,\frac{1}{2}\right]^d\right):\ \Delta\cap\Gamma\ne\varnothing\right\}. \tag{47}$$

Then $\{\hat{Q}_{\Delta_\nu}\}_{\nu\in\mathcal{N}}$ is a covering of $\Gamma$. Denote by $Q^*_{\Delta_\nu}$ the homothetic transform of $\hat{Q}_{\Delta_\nu}$ with respect to its center with coefficient 3. Applying Theorem D, we get that there exists a subset $\mathcal{N}'\subset\mathcal{N}$ such that $\{\hat{Q}_{\Delta_\nu}\}_{\nu\in\mathcal{N}'}$ are pairwise non-overlapping and $\{Q^*_{\Delta_\nu}\}_{\nu\in\mathcal{N}'}$ is a covering of $\Gamma$. We claim that

$$\mathrm{card}\,\mathcal{N}' \underset{\mathfrak{Z}_*}{\gtrsim} 2^{\theta\hat{k}t}(\hat{k}t)^{-\gamma}\tau^{-1}(\hat{k}t). \tag{48}$$

Indeed,

$$\mathrm{card}\,\mathcal{N}'\cdot 2^{-\theta\hat{k}t}(\hat{k}t)^\gamma\tau(\hat{k}t) \overset{(5)}{=} \mathrm{card}\,\mathcal{N}'\cdot h(2^{-\hat{k}t}) \overset{(3),(16)}{\underset{\mathfrak{Z}_*}{\asymp}} \sum_{\nu\in\mathcal{N}'}\mu(Q^*_{\Delta_\nu}) \geqslant \mu(\Gamma) \underset{\mathfrak{Z}_*}{\asymp} 1.$$

We take $\{c_{\Delta_\nu}\psi_{\Delta_\nu}\}_{\nu\in\mathcal{N}'}$ as the desired function set. It remains to apply Lemma 3 with $m=\hat{k}t$ and (48). □

Let us prove Proposition 2. Since $\beta-\delta+\theta\left(\frac{1}{q}-\frac{1}{p}\right)=0$ and $\beta_v=\frac{d-\theta}{q}$, then

$$\beta_g = r-\frac{d}{p}+\frac{\theta}{p}. \tag{49}$$

Let $t\in\mathbb{N}$ be sufficiently large, and let $\Delta\in\Xi_{\hat{k}t}\left([-1/2,\,1/2]^d\right)$, $\Delta\cap\Gamma\ne\varnothing$. For each $s\in\mathbb{Z}_+$ we set

$$\{\Delta_{s,i}\}_{i\in J_s} = \bigl\{\Delta'\in\Xi_{\hat{k}(t+s)}\bigl([-1/2,\,1/2]^d\bigr):\ \Delta'\subset\hat{Q}_\Delta,\ \ \Delta'\cap\Gamma\ne\varnothing\bigr\}. \tag{50}$$

Let

$$f_\Delta(x) = \sum_{s=0}^t\sum_{i\in J_s}\psi_{\Delta_{s,i}}(x),$$

where functions $\psi_{\Delta_{s,i}}$ are defined by formula similar to (42).

There are a number $t_0=t_0(\mathfrak{Z}_*)$ and a cube $\Delta_0\in\Xi_{\hat{k}(t-t_0)}\left([-1/2,\,1/2]^d\right)$ such that $\Delta\subset\Delta_0$, $\Gamma\cap\Delta_0\ne\varnothing$ and $\mathrm{supp}\, f_\Delta\subset\hat{Q}_{\Delta_0}$.

Let $l \in \mathbb{Z}_+$, $x \in \hat{E}_l(\Delta_0)\backslash\hat{E}_{l+1}(\Delta_0)$ (see (37) with $m = \hat{k}(t-t_0)$). Then $\mathrm{dist}_{|\cdot|}(x, \Gamma) \underset{\mathfrak{Z}_*}{\asymp} 2^{-\hat{k}(t+l)}$. We estimate $\left|\frac{\nabla^r f_\Delta(x)}{g(x)}\right|$ from above. If $x \in \mathrm{supp}\, \psi_{\Delta_{s,i}}$ for some $i \in J_s$, then

$$\left|\frac{\nabla^r \psi_{\Delta_{s,i}}(x)}{g(x)}\right| \overset{(4),(6),(37),(50)}{\underset{\mathfrak{Z}_*}{\lesssim}} 2^{-\beta_g \hat{k}(t+l)} (\hat{k}(t+l))^{\alpha_g} \rho_g^{-1}(\hat{k}(t+l)) \cdot 2^{r\hat{k}(t+s)}.$$

Moreover, by (50) we get $s \leqslant l + s_0$ with $s_0 = s_0(\mathfrak{Z}_*)$. Since $\mathrm{supp}\, \psi_{\Delta_{s,i}} \overset{(43)}{\subset} \hat{Q}_{\Delta_{s,i}}$, by the definition of $\hat{Q}_{\Delta_{s,i}}$ it follows that for any $x \in \hat{Q}_{\Delta_0}$ the inequality $\mathrm{card}\,\{i \in J_s : x \in \mathrm{supp}\, \psi_{\Delta_{s,i}}\} \underset{\mathfrak{Z}_*}{\lesssim} 1$ holds. Hence, for $l \leqslant t - s_0$

$$\left|\frac{\nabla^r f_\Delta(x)}{g(x)}\right| \underset{\mathfrak{Z}_*}{\lesssim} \sum_{s=0}^{l+s_0} 2^{-\beta_g \hat{k}(t+l)} (\hat{k}(t+l))^{\alpha_g} \rho_g^{-1}(\hat{k}(t+l)) \cdot 2^{r\hat{k}(t+s)} \overset{(15)}{\underset{\mathfrak{Z}_*}{\lesssim}}$$

$$\lesssim 2^{(r-\beta_g)\hat{k}(t+l)} (\hat{k}t)^{\alpha_g} \rho_g^{-1}(\hat{k}t),$$

and for $l > t - s_0$

$$\left|\frac{\nabla^r f_\Delta(x)}{g(x)}\right| \underset{\mathfrak{Z}_*}{\lesssim} \sum_{s=0}^{t} 2^{-\beta_g \hat{k}(t+l)} (\hat{k}(t+l))^{\alpha_g} \rho_g^{-1}(\hat{k}(t+l)) \cdot 2^{r\hat{k}(t+s)} \overset{(15)}{\underset{\mathfrak{Z}_*}{\lesssim}}$$

$$\lesssim 2^{-\beta_g \hat{k}(t+l)} (\hat{k}(t+l))^{\alpha_g} \rho_g^{-1}(\hat{k}(t+l)) \cdot 2^{2r\hat{k}t}.$$

This yields that

$$\left\|\frac{\nabla^r f_\Delta}{g}\right\|^p_{L_p(\Omega)} \overset{(38)}{=} \sum_{l=0}^{\infty} \left\|\frac{\nabla^r f_\Delta}{g}\right\|^p_{L_p(\hat{E}_l(\Delta_0)\backslash\hat{E}_{l+1}(\Delta_0))} \overset{(39)}{\underset{\mathfrak{Z}_*}{\lesssim}}$$

$$\lesssim \sum_{l=0}^{t-s_0} 2^{p(r-\beta_g)\hat{k}(t+l)} (\hat{k}t)^{p\alpha_g} \rho_g^{-p}(\hat{k}t) \cdot 2^{-\hat{k}td-(d-\theta)\hat{k}l} \frac{(\hat{k}t)^\gamma \tau(\hat{k}t)}{(\hat{k}(t+l))^\gamma \tau(\hat{k}(t+l))} +$$

$$+ \sum_{l=t-s_0+1}^{\infty} 2^{-p\beta_g \hat{k}(t+l)} (\hat{k}(t+l))^{p\alpha_g} \rho_g^{-p}(\hat{k}(t+l)) \cdot 2^{2r\hat{k}tp} \cdot 2^{-\hat{k}td-(d-\theta)\hat{k}l} \frac{(\hat{k}t)^\gamma \tau(\hat{k}t)}{(\hat{k}(t+l))^\gamma \tau(\hat{k}(t+l))} \overset{(49)}{\underset{\mathfrak{Z}_*}{\lesssim}}$$

$$\lesssim 2^{-\hat{k}t\theta} (\hat{k}t)^{\alpha_g p+1} \rho_g^{-p}(\hat{k}t).$$

Thus,

$$\left\|\frac{\nabla^r f_\Delta}{g}\right\|_{L_p(\Omega)} \underset{\mathfrak{Z}_*}{\lesssim} 2^{-\frac{\hat{k}t\theta}{p}} (\hat{k}t)^{\alpha_g + \frac{1}{p}} \rho_g^{-1}(\hat{k}t). \tag{51}$$

Let us estimate $\|f_\Delta\|_{L_{q,v}(\Omega)}$ from below. Let $x \in E_l(\Delta)\backslash\hat{E}_{l+1}(\Delta)$. Then $\mathrm{dist}_{|\cdot|}(x, \Gamma) \overset{(37)}{\underset{\mathfrak{Z}_*}{\asymp}} 2^{-\hat{k}(t+l)}$ and there exists $l_0 = l_0(\mathfrak{Z}_*)$ such that for $0 \leqslant s \leqslant l - l_0$ there exists $i_s \in J_s$

such that $x \in \tilde{Q}_{\Delta_{s,i_s}}$. (Indeed, since $x \in Q_\Delta$ by (37), there exists a point $y \in \Gamma \cap \hat{Q}_\Delta$ such that $|x-y| \underset{\mathfrak{Z}_*}{\asymp} 2^{-\hat{k}(t+l)}$. We choose a cube $\Delta_{s,i_s}$ that contains the point $y$. By the definition of the cube $Q_{\Delta_{s,i_s}}$, we have $\hat{x}_{\Delta_{s,i_s}} \in \Delta_{s,i_s}$; hence, $|y - \hat{x}_{\Delta_{s,i_s}}| \overset{(50)}{\leqslant} 2^{-\hat{k}(t+s)}$. Therefore, $|x - \hat{x}_{\Delta_{s,i_s}}| \leqslant |x-y| + |y - \hat{x}_{\Delta_{s,i_s}}| \leqslant c(\mathfrak{Z}_*)2^{-\hat{k}(t+l)} + 2^{-\hat{k}(t+s)}$ for some $c(\mathfrak{Z}_*) > 0$. It remains to apply (36), (50) and the inequality $s \leqslant l - l_0$.) Hence, for $\frac{t}{2} \leqslant l \leqslant t$ we have $|f_\Delta(x)| \underset{\mathfrak{Z}_*}{\overset{(43)}{\gtrsim}} t$. Consequently,

$$\|f_\Delta\|^q_{L_{q,v}(\Omega)} \geqslant \sum_{t/2\leqslant l\leqslant t} \|f_\Delta\|^q_{L_{q,v}(E_l(\Delta)\setminus \hat{E}_{l+1}(\Delta))} \underset{\mathfrak{Z}_*}{\overset{(4),(6),(37),(41)}{\gtrsim}}$$

$$\gtrsim \sum_{t/2\leqslant l\leqslant t} t^q \cdot 2^{\beta_v q\hat{k}(t+l)}(\hat{k}(t+l))^{-\alpha_v q}\rho_v^q(\hat{k}(t+l)) \cdot 2^{-\hat{k}td-(d-\theta)\hat{k}l} \frac{(\hat{k}t)^\gamma \tau(\hat{k}t)}{(\hat{k}(t+l))^\gamma \tau(\hat{k}(t+l))} \underset{\mathfrak{Z}_*}{\overset{(8)}{\gtrsim}}$$

$$\gtrsim 2^{-\theta\hat{k}t}(\hat{k}t)^{-\alpha_v q+q+1}\rho_v^q(\hat{k}t);$$

i.e.,

$$\|f_\Delta\|_{L_{q,v}(\Omega)} \underset{\mathfrak{Z}_*}{\gtrsim} 2^{-\frac{\theta\hat{k}t}{q}}(\hat{k}t)^{-\alpha_v+1+\frac{1}{q}}\rho_v(\hat{k}t). \qquad (52)$$

**Proof of Proposition 2.** Let the set of cubes $\{\Delta_\nu\}_{\nu\in\mathcal{N}}$ be defined by formula (47), and let $F_{\Delta_\nu} = c_{\Delta_\nu} f_{\Delta_\nu}$, with $c_{\Delta_\nu}$ such that $\left\|\frac{\nabla^r F_{\Delta_\nu}}{g}\right\|_{L_p(\Omega)} = 1$. From (51) and (52) it follows that

$$\|F_{\Delta_\nu}\|_{L_{q,v}(\Omega)} \underset{\mathfrak{Z}_*}{\gtrsim} 2^{-\theta\left(\frac{1}{q}-\frac{1}{p}\right)\hat{k}t}(\hat{k}t)^{-\alpha+\frac{1}{q}+1-\frac{1}{p}}\rho(\hat{k}t).$$

Further, $\operatorname{supp} F_{\Delta_\nu} = \operatorname{supp} f_{\Delta_\nu} \subset \hat{Q}_{(\Delta_\nu)_0}$ and $\operatorname{diam} \hat{Q}_{(\Delta_\nu)_0} \underset{\mathfrak{Z}_*}{\asymp} 2^{-\hat{k}t}$. We apply Theorem D to the covering $\{\hat{Q}_{(\Delta_\nu)_0}\}_{\nu\in\mathcal{N}}$ of the set $\Gamma$ and argue similarly as in the proof of Proposition 1. $\square$

**Remark 5.** *Let $\beta_v = \frac{d-\theta}{q}$, $\beta - \delta + \theta\left(\frac{1}{q}-\frac{1}{p}\right)_+ = 0$. In addition, let $\alpha < \frac{1}{q}$ in the case $1 < p < q < \infty$, and let $\alpha < 1 + (1-\gamma)\left(\frac{1}{q}-\frac{1}{p}\right)$ in the case $p \geqslant q$. Then Propositions 1 and 2 hold; it implies that $\vartheta_n(W^r_{p,g}(\Omega), L_{q,v}(\Omega)) = \infty$ for any $n \in \mathbb{Z}_+$. In particular, if we take $\vartheta_n = d_n$, then we get that the deviation of $W^r_{p,g}(\Omega)$ from any finite-dimensional subspace is infinite.*

**REFERENCES**


[1] M.S. Aitenova, L.K. Kusainova, "On the asymptotics of the distribution of approximation numbers of embeddings of weighted Sobolev classes. I", *Mat. Zh.*, **2**:1 (2002), 3–9.

[2] M.S. Aitenova, L.K. Kusainova, “On the asymptotics of the distribution of approximation numbers of embeddings of weighted Sobolev classes. II”, *Mat. Zh.*, **2**:2 (2002), 7–14.

[3] K. F. Andersen and H. P. Heinig, Weighted norm inequalities for certain integral operators, SIAM J. Math. Anal. **14**, 834–844 (1983).

[4] G. Bennett, Some elementary inequalities. III, Quart. J. Math. Oxford Ser. (2) **42**, 149–174 (1991).

[5] O.V. Besov, “Kolmogorov widths of Sobolev classes on an irregular domain”, *Proc. Steklov Inst. Math.*, **280** (2013), 34-45.

[6] O.V. Besov, V.P. Il’in, S.M. Nikol’skii, *Integral representations of functions, and imbedding theorems.* “Nauka”, Moscow, 1996. [Winston, Washington DC; Wiley, New York, 1979].

[7] M.Sh. Birman and M.Z. Solomyak, “Piecewise polynomial approximations of functions of classes $W_p^\alpha$”, *Mat. Sb.* **73**:3 (1967), 331–355 [Russian]; transl. in *Math USSR Sb.*, **2**:3 (1967), 295–317.

[8] I.V. Boykov, “Approximation of some classes of functions by local splines”, *Comput. Math. Math. Phys.*, **38**:1 (1998), 21–29.

[9] I.V. Boykov, “Optimal approximation and Kolmogorov widths estimates for certain singular classes related to equations of mathematical physics”, arXiv:1303.0416v1.

[10] M. Bricchi, “Existence and properties of $h$-sets”, *Georgian Mathematical Journal*, **9**:1 (2002), 13—32.

[11] M. Christ, “A $T(b)$ theorem with remarks on analytic capacity and the Cauchy integral”, *Colloq. Math.* **60/61**:2 (1990), 601–628.

[12] R.A. DeVore, R.C. Sharpley, S.D. Riemenschneider, “$n$-widths for $C_p^\alpha$ spaces”, *Anniversary volume on approximation theory and functional analysis (Oberwolfach, 1983)*, 213–222, Internat. Schriftenreihe Numer. Math., **65**, Birkhäuser, Basel, 1984.

[13] D.E. Edmunds, W.D. Evans, *Hardy Operators, Function Spaces and Embeddings.* Springer-Verlag, Berlin, 2004.

[14] D.E. Edmunds, H. Triebel, *Function spaces, entropy numbers, differential operators.* Cambridge Tracts in Mathematics, **120** (1996). Cambridge University Press.

[15] A. El Kolli, “$n$-ième épaisseur dans les espaces de Sobolev”, *J. Approx. Theory*, **10** (1974), 268–294.

[16] W.D. Evans, D.J. Harris, J. Lang, “The approximation numbers of Hardy-type operators on trees”, *Proc. London Math. Soc.* **(3) 83**:2 (2001), 390–418.

[17] W. D. Evans, D. J. Harris, and L. Pick, Weighted Hardy and Poincaré inequalities on trees, J. London Math. Soc. **52**, 121–136 (1995).

[18] H. P. Heinig, Weighted norm inequalities for certain integral operators, II, Proc. AMS. **95**, 387–395 (1985).

[19] B.S. Kashin, "The widths of certain finite-dimensional sets and classes of smooth functions", *Math. USSR-Izv.*, **11**:2 (1977), 317–333.

[20] L.D. Kudryavtsev and S.M. Nikol'skii, "Spaces of differentiable functions of several variables and imbedding theorems," in Analysis-3 (VINITI, Moscow, 1988), Itogi Nauki Tekh., Ser.: Sovrem. Probl. Mat., Fundam. Napravl. 26, pp. 5–157; Engl. transl. in Analysis III (Springer, Berlin, 1991), Encycl. Math. Sci. 26, pp. 1–140.

[21] A. Kufner, *Weighted Sobolev spaces.* Teubner-Texte Math., 31. Leipzig: Teubner, 1980.

[22] G. Leoni, *A first Course in Sobolev Spaces.* Graduate studies in Mathematics, vol. 105. AMS, Providence, Rhode Island, 2009.

[23] M.A. Lifshits, "Bounds for entropy numbers for some critical operators", *Trans. Amer. Math. Soc.*, **364**:4 (2012), 1797–1813.

[24] M.A. Lifshits, W. Linde, "Compactness properties of weighted summation operators on trees", *Studia Math.*, **202**:1 (2011), 17–47.

[25] M.A. Lifshits, W. Linde, "Compactness properties of weighted summation operators on trees — the critical case", *Studia Math.*, **206**:1 (2011), 75–96.

[26] P.I. Lizorkin, M. Otelbaev, "Estimates of approximate numbers of the imbedding operators for spaces of Sobolev type with weights", *Trudy Mat. Inst. Steklova*, **170** (1984), 213–232 [*Proc. Steklov Inst. Math.*, **170** (1987), 245–266].

[27] K. Mynbaev, M. Otelbaev, *Weighted function spaces and the spectrum of differential operators.* Nauka, Moscow, 1988.

[28] M.O. Otelbaev, "Estimates of the diameters in the sense of Kolmogorov for a class of weighted spaces", *Dokl. Akad. Nauk SSSR*, **235**:6 (1977), 1270–1273 [Soviet Math. Dokl.].

[29] A. Pinkus, *n-widths in approximation theory.* Berlin: Springer, 1985.

[30] Yu.G. Reshetnyak, "Integral representations of differentiable functions in domains with a nonsmooth boundary", *Sibirsk. Mat. Zh.*, **21**:6 (1980), 108–116 (in Russian).

[31] Yu.G. Reshetnyak, "A remark on integral representations of differentiable functions of several variables", *Sibirsk. Mat. Zh.*, **25**:5 (1984), 198–200 (in Russian).

[32] M. Solomyak, "On approximation of functions from Sobolev spaces on metric graphs", *J. Approx. Theory*, **121**:2 (2003), 199–219.

[33] V.M. Tikhomirov, *Some questions in approximation theory.* Izdat. Moskov. Univ., Moscow, 1976 [in Russian].

[34] V.M. Tikhomirov, "Theory of approximations". In: *Current problems in mathematics. Fundamental directions.* vol. 14. (*Itogi Nauki i Tekhniki*) (Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1987), pp. 103–260 [Encycl. Math. Sci. vol. 14, 1990, pp. 93–243].

[35] V.M. Tikhomirov, "Diameters of sets in functional spaces and the theory of best approximations", *Russian Math. Surveys*, **15**:3 (1960), 75–111.

[36] H. Triebel, *Interpolation theory, function spaces, differential operators* (North-Holland Mathematical Library, 18, North-Holland Publishing Co., Amsterdam–New York, 1978; Mir, Moscow, 1980).

[37] H. Triebel, *Theory of function spaces III.* Birkhäuser Verlag, Basel, 2006.

[38] H. Triebel, "Entropy and approximation numbers of limiting embeddings, an approach via Hardy inequalities and quadratic forms", *J. Approx. Theory*, **164**:1 (2012), 31–46.

[39] A.A. Vasil'eva, "Widths of weighted Sobolev classes on a John domain", *Proc. Steklov Inst. Math.*, **280** (2013), 91–119.

[40] A.A. Vasil'eva, "An embedding theorem for weighted Sobolev classes on a John domain: case of weights that are functions of a distance to a certain $h$-set", *Russ. J. Math. Phys.*, **20**:3 (2013), 360–373.

[41] A.A. Vasil'eva, "Embedding theorem for weighted Sobolev classes on a John domain with weights that are functions of the distance to some $h$-set. II", *Russ. J. Math. Phys.*, **21**:1 (2014), 112–122.

[42] A.A. Vasil'eva, "Estimates for norms of two-weighted summation operators on a tree under some conditions on weights", arXiv.org:1311.0375.

[43] A.A. Vasil'eva, "Widths of function classes on sets with tree-like structure", arxiv.org:1312.7231.

[44] A.A. Vasil'eva, "Kolmogorov and linear widths of the weighted Besov classes with singularity at the origin", *J. Appr. Theory*, **167** (2013), 1–41.

[45] A.A. Vasil'eva, "Some sufficient conditions for embedding a weighted Sobolev class on a John domain", *Siberian Math. J.*, to appear.

[46] A.A. Vasil'eva, "Widths of weighted Sobolev classes on a John domain: strong singularity at a point", *Rev. Mat. Compl.*, **27**:1 (2014), 167–212.